Fractional processes: New limit theorems and statistical inference

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Outline

- Fractional Brownian motion and its properties
- Limit theorems and statistical inference
- Fractional stable motion and its properties
- Some probabilistic results
- Estimation methods and asymptotic results

Fractional Brownian motion

- *Definition:* The scaled fBm $Z_t = \sigma B_t^H$ with Hurst parameter $H \in (0,1)$ is a zero mean Gaussian process characterised by

$$\mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

- *Properties:* fBm has **stationary increments** and it is **self-similar** with index H, i.e.

$$(Z_{at})_{t\in\mathbb{R}_+} \stackrel{d}{=} (a^H Z_t)_{t\in\mathbb{R}_+}, \qquad a > 0$$

- *Smoothness:* It holds that

$$Z \in C^{H-}([0,1]) \qquad \text{almost surely}$$

There exist various ways of representing a fBm through a Brownian motion W.

(i) *Mandelbrot-van Ness representation:*

$$Z_t = \mathrm{const} \cdot \int_{\mathbb{R}} \left\{ (t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \right\} dW_s$$

where $x_+ = \max\{0, x\}$ and W is a Brownian motion.

(ii) *Harmonizable representation:*

$$Z_t = \mathrm{const} \cdot \int_{\mathbb{R}} \frac{\exp(its) - 1}{|s|^{H+1/2}} \, d\mathbb{W}_s$$

where $\mathbb{W}$ is a certain complex Gaussian measure.

We now demonstrate optimal rates of convergence for statistical estimation of

$$\theta = (\sigma, H)$$

in **low** and **high frequency** setting.

Theorem: [Dahlhaus (89), Coeurjolly and Istas (01)]

(i) *Low frequency:* The optimal estimation rate for θ is

$$\left(\sqrt{n}, \sqrt{n}\right)$$

(ii) *High frequency:* The optimal estimation rate for θ is

$$\left(\sqrt{n}/\log n, \sqrt{n}\right)$$

If σ is known the optimal estimation rate for H is $\sqrt{n}\log n$.

Theorem: [Breuer and Major (83), Taqqu (79)] Define

$$V(Z;p)_n = n^{-1+pH} \sum_{i=1}^{n} |\Delta_i^n Z|^p \qquad \text{with} \qquad \Delta_i^n Z = Z_{i/n} - Z_{(i-1)/n}$$

(i) *Law of large numbers:* As $n \to \infty$

$$V(Z;p)_n \xrightarrow{\mathbb{P}} \sigma^p m_p \qquad \text{where} \qquad m_p := \mathbb{E}[|\mathcal{N}(0,1)|^p]$$

(ii) *Central limit theorem:* When $H \in (0, 3/4)$ we obtain a central limit theorem

$$\sqrt{n}\,(V(Z;p)_n - \sigma^p m_p) \xrightarrow{d} \mathcal{N}(0, a^2_{\sigma,H})$$

If $H \in (3/4, 1)$ we have that

$$n^{2-2H}\,(V(Z;p)_n - \sigma^p m_p) \xrightarrow{d} \text{Rosenblatt rv}$$

- *Estimation of H:* We use the ratio statistic

$$R(p)_n := \frac{\sum_{i=2}^{n} \left| Z_{i/n} - Z_{(i-2)/n} \right|^p}{\sum_{i=1}^{n} \left| Z_{i/n} - Z_{(i-1)/n} \right|^p} \xrightarrow{\mathbb{P}} 2^{pH}$$

Hence, the obvious estimator of the parameter H is

$$\widehat{H}(p)_n := p^{-1} \log_2 R(p)_n$$

- *Estimation of σ^2:* Now, we define the plug-in estimator

$$\widehat{\sigma^2}(p)_n := n^{-1+2\widehat{H}(p)_n} \sum_{i=1}^{n} |\Delta_i^n Z|^2 \xrightarrow{\mathbb{P}} \sigma^2$$

- *Convergence rates:* When $H \in (0, 3/4)$ the convergence rate is

$$(\sqrt{n}/\log n, \sqrt{n})$$

Related work

- Brouste and Fukasawa (17) show *local asymptotic normality* of the fBm in high frequency regime.
- Bardet and Surgailis (13), Lebovits and P. (17) investigate non-parametric estimation of the Hurst function for the *multifractional Brownian motion*.
- Barndorff-Nielsen, Corcuera and P. (09) investigate statistical inference for *integrals wrt Gaussian processes* in high frequency regime.

Theorem: [Chigansky and Kleptsyna (18)] Let K denote the covariance kernel of a standard fBm. Define the operator

$$\mathcal{K}f(t) := \int_0^1 K(s,t)f(s)ds$$

Then $\mathcal{K}$ admits a countable spectrum $(\lambda_n)_{n\geq 1}$ and

$$\lambda_n = \sin(\pi H)\Gamma(2H+1)\nu_n^{-2H-1}$$

where

$$\nu_n = \pi(n-1/2) + \pi(1-2H)/4 + \arcsin\left(l_H/\sqrt{1+l_H^2}\right) + O(1/n)$$

Moreover, it holds

$$\mathbb{P}\left(\|B^H\|_2 \leq \varepsilon\right) \sim C_H \varepsilon^{\gamma(l_H)} \exp\left(-\beta_H \varepsilon^{-1/H}\right) \qquad \text{as } \varepsilon \to 0$$

Question

How large is the class of *self-similar stable* processes with *stationary increments*?

Theorem: [Rosinski (95)] Every stationary symmetric α-stable process $(Z_t)_{t\in\mathbb{R}}$ possess a unique (in distribution) decomposition

$$Z = Z^1 + Z^2 + Z^3 \qquad \text{where}$$

- Z^1 is a *mixed moving average* process

$$Z_t^1 = \int_A \int_{\mathbb{R}} g(x, t-u) N(dx, du)$$

  where N is a symmetric α-stable measure on $A \times \mathbb{R}$.
- Z^2 is a *harmonizable* process

$$Z_t^2 = \int_{\mathbb{R}} \exp(itu) \overline{N}(du)$$

  where $\overline{N}$ is a complex isotropic α-stable measure.
- Z^3 is generated by a certain *conservative flow*.

Fractional stable motion

- *Definition:* The fractional stable motion $(X_t)_{t\in\mathbb{R}}$ (fsm) is defined by

$$X_t = \int_{\mathbb{R}} \left\{ (t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha} \right\} dL_s$$

where $H \in (0,1)$ and $(L_t)_{t\in\mathbb{R}}$ is a SαS Lévy motion with $\alpha \in (0,2)$ and scale parameter $\sigma > 0$.

- *Stable Lévy processes:* A SαS Lévy motion $(L_t)_{t\in\mathbb{R}}$ with scale parameter $\sigma > 0$ is fully characterised by *stationary* and *independent* increments, and by the characteristic function

$$\mathbb{E}[\exp(itL_1)] = \exp(-\sigma^\alpha |t|^\alpha) \qquad t \in \mathbb{R}$$

- *Probabilistic structure:* For $H \neq 1/\alpha$ the fsm is neither Markovian nor a Lévy process.

Fractional stable motion: Properties

- *Marginal distributions:* All marginal distributions of $(X_t)_{t\in\mathbb{R}}$ are SαS. In particular

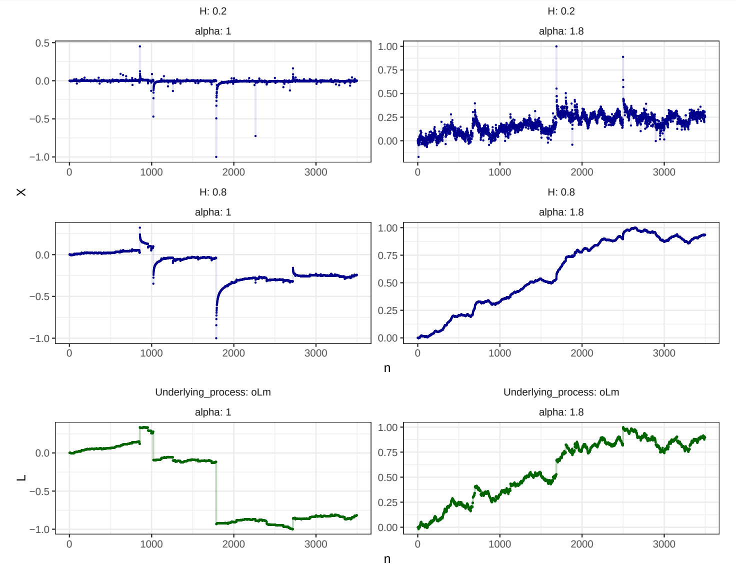
$$\mathbb{E}[\exp(iuX_t)] = \exp(-\sigma^\alpha c_t |t|^\alpha), \quad c_t = \int_{\mathbb{R}} \left|(t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha}\right|^\alpha ds$$

- *Self-similarity:* The process $(X_t)_{t\in\mathbb{R}}$ is self-similar with index $H \in (0,1)$.

- *Path properties I:* If $H - 1/\alpha > 0$ the process $(X_t)_{t\in\mathbb{R}}$ is locally Hölder continuous of any order up to $H - 1/\alpha$.

- *Path properties II:* If $H - 1/\alpha < 0$ the process $(X_t)_{t\in\mathbb{R}}$ has unbounded path on compact intervals.

Paths behaviour of a fractional stable motion



- *Higher order increments:* We denote by $\Delta_{i,k}^n X$ the kth order increment of X at stage i/n, i.e.

$$\Delta_{i,k}^n X := \sum_{j=0}^{k} (-1)^j \binom{k}{j} X_{(i-j)/n} \qquad i \geq k$$

For example, $\Delta_{i,1}^n X = X_{i/n} - X_{(i-1)/n}$.

- *Main statistics:* Our probabilistic tools are statistics of the form

$$V(f;k)_n := a_n \sum_{i=k}^{n} f\left(b_n \Delta_{i,k}^n X\right)$$

where a_n and b_n are certain deterministic sequences.

- The most useful examples in statistics are:

$$f_1(x) = |x|^p \quad p > 0 \qquad \text{(power variation)}$$

$$f_2(x) = |x|^{-p} \quad p \in (0,1) \qquad \text{(negative power variation)}$$

$$f_3(x) = \cos(ux) \text{ or } \sin(ux) \qquad \text{(empirical characteristic function)}$$

$$f_4(x) = 1_{(-\infty,u]}(x) \qquad \text{(empirical distribution function)}$$

$$f_5(x) = \log(|x|)1_{\{x\neq 0\}} \qquad \text{(logarithmic function)}$$

- The power variation case $f_1(x) = |x|^p$ has been investigated in Basse-O'Connor, Lachieze-Rey and P. (17). Previously, no theoretical results have been established outside this class of functions.

Law of large numbers: Notations

- When Y is a symmetric α-stable random variable with scale parameter $\rho > 0$ we write

$$Y \sim S\alpha S(\rho)$$

- We introduce the function

$$h_k(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x-j)_+^{H-1/\alpha} \qquad x \in \mathbb{R}$$

- $(T_m)_{m\geq 1}$ denote the jump times of L. We also introduce a sequence of random variables

$$(U_m)_{m\geq 1} \text{ i.i.d. } \sim \mathcal{U}(0,1)$$

which is independent of L.

Theorem: Assume that $\alpha \in (0,2)$ and $H - 1/\alpha > 0$.

(i) When $f \in C^p(\mathbb{R})$ for some $p > \alpha$ and $f(0) = .. = f^{[p]}(0) = 0$, then it holds

$$\sum_{i=k}^{n} f\left(n^{H-1/\alpha}\Delta_{i,k}^n X\right) \xrightarrow{d} \sum_{m:\, T_m \in [0,1]} \sum_{l=0}^{\infty} f\left(\Delta L_{T_m} h_k(l + U_m)\right)$$

(ii) Assume that $\mathbb{E}[|f(L_1)|] < \infty$. Then it holds

$$\frac{1}{n}\sum_{i=k}^{n} f\left(n^{H}\Delta_{i,k}^n X\right) \xrightarrow{\mathbb{P}} \mathbb{E}[f(S)], \quad S \sim S\alpha S(\sigma\|h_k\|_\alpha)$$

- *Estimation of H:* For $p \in (0,1)$, we use the ratio statistic

$$R(-p)_n := \frac{\sum_{i=2}^{n} \left| X_{i/n} - X_{(i-2)/n} \right|^{-p}}{\sum_{i=1}^{n} \left| X_{i/n} - X_{(i-1)/n} \right|^{-p}} \xrightarrow{\mathbb{P}} 2^{-pH}$$

Hence, the obvious estimator of the parameter H is

$$\widehat{H}(-p)_n := -p^{-1} \log_2 R(-p)_n$$

- *Estimation of (σ, α):* It holds that

$$\varphi_n(u) := \frac{1}{n} \sum_{i=k}^{n} \cos\left(u n^H \Delta_{i,k}^n X\right) \xrightarrow{\mathbb{P}} \varphi(u) := \exp\left(-|u\sigma \|h_k\|_\alpha|^\alpha\right)$$

The latter consistency result can be used to estimate (σ, α).

Central limit theorem: Assumptions

- *Appell rank:* To determine the weak limit theory associated with case (ii), we introduce the function

$$\Phi_\rho(x) := \mathbb{E}[f(S+x)] - \mathbb{E}[f(S)], \qquad S \sim S\alpha S(\rho)$$

  We define the *Appell rank* $m_\rho^\star$ by

$$m_\rho^\star := \min\{r \geq 1 : \ \Phi_\rho^{(r)}(0) \neq 0\}$$

- *Assumptions on Φ:* We assume that the map $(\rho, x) \mapsto \Phi_\rho(x)$ is in $C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ along with some boundedness and growth conditions on derivatives.

- *Notation:*

$$V(f;k)_n := \frac{1}{n} \sum_{i=k}^{n} f\left(n^H \Delta_{i,k}^n X\right), \qquad V(f;k) := \mathbb{E}[f(S)]$$

Theorem: Assume that $L \sim S\alpha S(\sigma)$ and $\mathbb{E}[f^2(L_1)] < \infty$.

(i) When $k > H + 1/\alpha$ we obtain

$$\sqrt{n}\,(V(f;k)_n - V(f;k)) \xrightarrow{d} \mathcal{N}(0, v_f^2)$$

(ii) When $m^\star_{\rho_0} \geq 2$ for $\rho_0 = \sigma\|h_k\|_\alpha$ and $k < H + 1/\alpha$ we obtain

$$n^{\frac{\alpha(k-H)}{\alpha(k-H)+1}}\,(V(f;k)_n - V(f;k)) \xrightarrow{d} S_2 = \alpha(k-H)+1-\text{stable}.$$

(iii) When $m^\star_{\rho_0} = 1$ for $\rho_0 = \sigma\|h_k\|_\alpha$, $\alpha \in (1,2)$ and $k < H + 1 - 1/\alpha$:

$$n^{k-H}\,(V(f;k)_n - V(f;k)) \xrightarrow{d} S_1 = \alpha - \text{stable}.$$

Estimation for $H - 1/\alpha > 0$ in high frequency

- In the continuous setting $H - 1/\alpha > 0$ we necessarily have that $\alpha \in (1, 2)$. Hence, the statistics

$$\widehat{H}(p)_n \qquad \text{and} \qquad \varphi_n(u) = \varphi_n\left(\widehat{H}(p)_n, u\right)$$

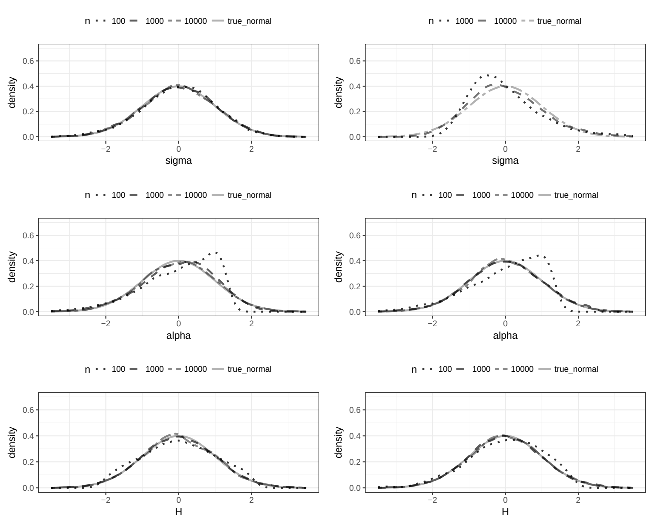
  are asymptotically normal if $p < 1/2$ and $k \geq 2$.

- To estimate the parameter (σ, α, H) it suffices to use $\widehat{H}(p)_n$ and $\varphi_n(u)$, $u \in \{1, 2\}$. By delta method we obtain asymptotically normal estimators with convergence rate

$$\left(\sqrt{n}/\log n, \sqrt{n}/\log n, \sqrt{n}\right)$$

- In the following simulation study the true parameter is $(\sigma, \alpha, H) = (0.3, 1.8, 0.8)$, $p = 0.4$, $k = 2$ and $n = 100, 1.000, 10.000$.

Numerical results



Estimation of $\theta = (\sigma, \alpha)$ in low frequency setting

- *Statistical problem:* We consider the fsm model

$$X_t = \int_{\mathbb{R}} \left\{ (t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha} \right\} dL_s$$

observed at low frequency $X_1, X_2, \ldots, X_n$ with known H.

- *Minimal contrast estimator:* To estimate the parameter $\theta = (\sigma, \alpha) \in \Theta = \mathbb{R}_+ \times (0, 2)$, we introduce

$$\varphi_n(u) := \frac{1}{n} \sum_{i=k}^{n} \cos\left(u \Delta_{i,k} X\right) \xrightarrow{a.s.} \varphi_\theta(u) := \exp\left(-|u\sigma \|h_k\|_\alpha|^\alpha\right)$$

For a positive weight function $w \in L^1(\mathbb{R}_+)$, the minimal contrast estimator is defined by

$$\widehat{\theta}_n \in \operatorname{argmin}_{\theta \in \Theta} \|\varphi_n - \varphi_\theta\|_{L^2_w} := \operatorname{argmin}_{\theta \in \Theta} \int_{\mathbb{R}_+} |\psi(u) - \varphi_\theta(u)|^2 w(u) du$$

Theorem: Assume that $k \geq 2$.

(i) It holds that $\widehat{\theta}_n \xrightarrow{a.s.} \theta$ and

$$\sqrt{n}\,(\varphi_n(u) - \varphi_\theta(u)) \xrightarrow{f.i.d.i.} G$$

where G is a certain Gaussian process on $\mathbb{R}_+$.

(ii) Define the function $F(\psi, \theta) := \|\psi - \varphi_\theta\|^2_{L^2_w}$. Then we obtain the central limit theorem

$$\sqrt{n}\left(\widehat{\theta}_n - \theta\right) \xrightarrow{d} 2\mathrm{Hess}_\theta F(\varphi_\theta, \theta)^{-1} \left(\langle \frac{\partial}{\partial \theta_j} \varphi_\theta, G \rangle_{L^2_w}\right)_{j=1,2}$$

Thank you very much for your attention!