Aspects on the law of large numbers

Allan Gut

Uppsala University

SMSA, Berlin — 22 February, 2017

How it began 300 years ago

$X_1, X_2, \ldots$ i.i.d. Bernoulli(p), $S_n = \sum_{k=1}^n X_k$. Then

$$\frac{S_n}{n} \xrightarrow{p} p \quad \text{as} \quad n \to \infty.$$

Took Bernoulli 20 years or so.

WLLN with finite variance

Chebyshev('s inequality); late 1800's

Weierstrass and WLLN

Every $u \in C[0,1]$ can be approximated by a polynomial with a uniform precision.

Define the Bernstein polynomial

$$u_n(x) = \sum_{k=0}^{n} u\Big(\frac{k}{n}\Big)\binom{n}{k} x^k (1-x)^{n-k},$$

let $X, X_1, X_2, \ldots$ be i.i.d. $\mathrm{Be}(x)$, set $Y_n = \frac{1}{n}\sum_{k=1}^n X_k$.

Then: $Y_n \xrightarrow{p} x$ (WLLN),

$$u_n(x) = E\, u(Y_n) \to u(x) \quad \text{as} \quad n \to \infty$$

via WLLN (center) and Chebyshev (tails)

Note More than existence — *construction* for all n!

Weierstrass = WLLN + Tjebyshev : –)

The St. Petersburg Game

I toss a coin repeatedly.

First head at trial n $\Longrightarrow$ you win $X = 2^n$ Euro (Ducates).

What is a fair price for playing?

$$P(X = 2^n) = \frac{1}{2^n} \quad \Longrightarrow \quad E\,X = +\infty.$$

Let $X_1, X_2, \ldots$ be gains, $S_n = \sum_{k=1}^n X_k$, $c > 0$.

$$\sum_{n=1}^{\infty} P(X_n > cn) \geq \sum_{n=1}^{\infty} \frac{2}{cn} = +\infty.$$

$$\Longrightarrow \limsup_{n\to\infty} \frac{X_n}{n} = +\infty \quad \Longrightarrow \quad \limsup_{n\to\infty} \frac{S_n}{n} = +\infty.$$

No hope for a SLLN.

A fair price seems impossible.

An alternative

Determine a fee such that a WLLN holds:

That is, find $\{b_n,\ n \geq 1\}$, such that

$$\frac{S_n}{b_n} \xrightarrow{p} 1 \quad \text{as} \quad n \to \infty.$$

Does there exist such a weak law?

We shall return to this problem soon, but first:

Does there exist general weak laws with $E\,X = +\infty$?

A general weak law

$X_1, X_2, \ldots$ independent, $S_n = \sum_{k=1}^n X_k$

$\{b_n,\ n \geq 1\}$ reals, $\nearrow +\infty$,

$Y_{k,n} = X_k I\{|X_k| \leq b_n\}$, $1 \leq k \leq n$, $n \geq 1$,

$S_n' = \sum_{k=1}^n Y_{k,n}$, $\mu_n = E\, S_n'$.

If

$$\sum_{k=1}^{n} P(|X_k| > b_n) \to 0, \qquad (1)$$

$$\frac{1}{b_n^2}\sum_{k=1}^{n} \mathrm{Var}\ Y_{k,n} \to 0, \qquad (2)$$

then

$$\frac{S_n - \mu_n}{b_n} \xrightarrow{p} 0. \qquad (3)$$

If, in addition,

$$\frac{\mu_n}{b_n} \to 0, \qquad (4)$$

then

$$\frac{S_n}{b_n} \xrightarrow{p} 0. \qquad (5)$$

Conversely, (3) + (4) $\Longrightarrow$ (1) + (2).

Proof of sufficiency

Truncated Chebyshev inequality
+ trivialities.

Note

Conditions not only Tailor made

Also necessary!

Proof by "slicing device"

$$\begin{aligned}
\frac{1}{n^2} n E(X^2 I\{|X| \le n\}) &= \frac{1}{n} E(X^2 I\{|X| \le n\}) \\
&= \frac{1}{n} \sum_{k=1}^{n} E(X^2 I\{k-1 < |X| \le k\}) \\
&\le \frac{1}{n} \sum_{k=1}^{n} k^2 P(k-1 < |X| \le k) \\
&\le \frac{1}{n} \sum_{k=1}^{n} \Big(\sum_{j=1}^{k} 2j\Big) P(k-1 < |X| \le k) \\
&= \frac{1}{n} \sum_{j=1}^{n} 2j \sum_{k=j}^{n} P(k-1 < |X| \le k) \\
&= \frac{2}{n} \sum_{j=1}^{n} j\, P(j-1 < |X| \le n) \\
&\le \frac{4}{n} \sum_{j=0}^{n} j\, P(|X| > j) \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Kolmogorov–Feller with fatter tails

Theorem — G(2004)

- ♣ $X, X_1, X_2, \ldots$ i.i.d.
- ♣ $\rho \in (0,1]$, $b \in \mathcal{RV}(1/\rho)$:
- ♣ $b_n = b(n)$, $n \ge 1$.

Then

$$\frac{S_n - nE\big(XI\{|X| \le b_n\}\big)}{b_n} \xrightarrow{p} 0,$$

if and only if

$$nP(|X| > b_n) \to 0.$$

Proof Second condition follows from first.

Once again, one condition suffices!

The Kolmogorov–Feller weak law

Question Finite mean necessary?

$X, X_1, X_2, \ldots$ i.i.d. Then

$$\frac{S_n - nE\big(XI\{|X| \le n\}\big)}{n} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty,$$

if and only if

$$nP(|X| > n) \to 0 \quad \text{as} \quad n \to \infty.$$

Note

$$nP(|X| > n) \to 0 \quad \not\Longleftarrow \quad E|X| < \infty\,.$$

Proof Second condition follows from first.

Thus .... One condition suffices!

An example with infinite mean

$X_1, X_2, \ldots$ i.i.d.

$$f(x) = \begin{cases} \frac{1}{2x^2}, & \text{for } |x| > 1, \\ 0, & \text{otherwise.} \end{cases}$$

Unfortunately ...

$$nP(|X| > n) = n\int_n^\infty \frac{1}{x^2}\,dx = 1;$$

the Kolmogorov–Feller law fails.
However, one checks that

$$\frac{S_n}{n} \xrightarrow{d} \text{Cauchy} \quad \text{as} \quad n \to \infty.$$

An example

Pareto again, $0 < \rho \le 1$

$$f(x) = \frac{1}{\rho |x|^{\rho+1}} \quad \text{for} \quad |x| > 1.$$

Then,

$$nP(|X| > (n\log n)^{1/\rho}) = \frac{1}{\log n} \to 0\,,$$

$$\Longrightarrow \quad \frac{S_n}{(n\log n)^{1/\rho}} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty.$$

Back to St. Petersburg

Recall:
I toss a coin repeatedly.

First head at trial $n \implies$ you win $X = 2^n$ SEK.

What is a fair price for playing?

$$P(X = 2^n) = \frac{1}{2^n} \implies E\,X = +\infty.$$

No hope for a SLLN. No fair price possible.

Variant
Determine a fee such that a WLLN holds.

That is, find $\{b_n,\ n \geq 1\}$, such that

$$\frac{S_n}{b_n} \xrightarrow{p} 1 \quad \text{as} \quad n \to \infty.$$

Solution

$$nP(X > n\log_2 n) = \cdots \leq \cdots \leq \frac{2}{\log_2 n} \to 0.$$

Thus,

$$\frac{S_n - nE\big(XI\{|X| \leq n\log_2 n\}\big)}{n\log_2 n} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty.$$

Moreover,

$$nE\big(XI\{|X| \leq n\log_2 n\}\big) \sim n\log_2 n,$$

$$\implies \quad \frac{S_n}{n\log_2 n} \xrightarrow{p} 1 \quad \text{as} \quad n \to \infty.$$

Convergence in distribution?

No, but for a subsequence:

Theorem (Martin-Löf, 1985)
Let $N = 2^n$. Then

$$\frac{S_N}{N} - n \xrightarrow{d} Z,$$

where

$$\varphi_Z(t) = \exp\Big\{\sum_{k=-\infty}^{-1}\big(\exp\{it2^k\} - 1 - it2^k\big)\cdot 2^{-k} + \sum_{k=0}^{\infty}\big(\exp\{it2^k\} - 1\big)\cdot 2^{-k}\Big\}.$$

The limit distribution is semistable in the sense of Lévy.
Hence, no limit distribution for the full sequence.

Convergence a.s. ?

$$\sum_{n=1}^{\infty} P(X_n > cn\log_2 n) \geq \sum_{n=1}^{\infty}\frac{2}{cn\log_2 n} = +\infty.$$

$$\implies \limsup_{n\to\infty}\frac{X_n}{n\log_2 n} = +\infty \implies \limsup_{n\to\infty}\frac{S_n}{n\log_2 n} = +\infty.$$

Also,

$$\liminf_{n\to\infty}\frac{S_n}{n\log_2 n} = 1.$$

The weak law cannot be strengthened to a strong law.

However, ...

Trimmed sums

$X_1, X_2, \ldots$ i.i.d. etc

$$M_n = \max_{1\leq k\leq n} X_k, \qquad S_n^{(1)} = S_n - M_n.$$

Limits??

Csörgő–Simons (1996), St. P:

$$\frac{S_n^{(1)}}{n\log_2 n} \xrightarrow{a.s.} 1 \quad \text{as} \quad n \to \infty.$$

No big surprise, since typical for domains of attraction:

S_n is large because some X_k is large.

More generally (joint with Anders M-L)

Max-trimming

$$\tau_n^+ = \#\{\text{obs} = M_n\} = \#\{k \leq n : X_k = M_n\}$$

$$M_n^* = \tau_n^+ \cdot M_n = \text{sum of all maximals.}$$

One can show: $\dfrac{M_n^*}{n\log_2 n} \xrightarrow{p} 0$ as $n \to \infty$,

$$\implies \quad \frac{S_n^*}{n\log_2 n} = \frac{S_n - M_n^*}{n\log_2 n} \xrightarrow{p} 1 \quad \text{as} \quad n \to \infty.$$

Problems

- ◇ a.s. convergence ?? ($\limsup \leq 1$ trivially)
- ◇ $\dfrac{S_N^*}{N} - n \xrightarrow{d}$ M-L analog

Min-trimming Similarly and/but easier.

An extension (joint with Anders M-L)

First head at trial $k \implies$ you win $sr^{(k-1)}$ SEK

$X, X_1, X_2, \ldots$ i.i.d., $s, r > 0$ arbitrary.

$$P(X = sr^{k-1}) = pq^{k-1}, \quad k = 1, 2, \ldots,$$

Analogous results.

Remarks

- The case $s = 1/p,\ r = q^{-1/\alpha},\ 0 < \alpha \le 1,$ has been treated earlier.
- $s = r = 2,\ p = q = 1/2,$ reduces to the classical case.
- The case $r < 1/q \implies$ finite mean, hence,

$$\frac{S_n}{n} \overset{a.s.}{\to} \frac{sp}{1 - rq} \quad \text{as} \quad n \to \infty \quad \ldots\ldots \quad \text{not interesting.}$$

The strong law

The stabilization of the relative frequences

Let A be an event. We perform independent repetitions.

$X_k = 1$ if round k is successful and 0 otherwise.

The relative frequency of successes $= \frac{1}{n}\sum_{k=1}^n X_k$.

WLLN tells us that $\frac{1}{n}\sum_{k=1}^n X_k \overset{p}{\to} P(A)$.

However, the stabilization of the relative frequencies corresponds to

$$\frac{1}{n}\sum_{k=1}^n X_k \overset{a.s.}{\to} P(A) \quad \text{as} \quad n \to \infty.$$

So ... what about a general strong law? SLLN?

The Kolmogorov strong law

$X_1, X_2, \ldots$ i.i.d. $S_n = \sum_{k=1}^n X_k,\ n \ge 1$.

- If $E|X| < \infty$ and $E\,X = \mu$, then

$$\frac{S_n}{n} \overset{a.s.}{\to} \mu \quad \text{as} \quad n \to \infty.$$

- If $\frac{S_n}{n} \overset{a.s.}{\to} c$, then

$$E|X| < \infty \quad \text{and} \quad c = E\,X$$

- If $E|X| = \infty$, then

$$\limsup_{n\to\infty} \frac{|S_n|}{n} = +\infty.$$

Proofs of a.s. convergence typically involve Borel–Cantelli sums.

Therefore . . .

Complete convergence

Definition $\{U_n,\ n \ge 1\}$ random variables.

U_n converges completely to the constant c as $n \to \infty$ iff

$$\sum_{n=1}^\infty P(|U_n - c| > \varepsilon) < \infty, \quad \text{for all} \quad \varepsilon > 0.$$

Notation: $U_n \overset{c.c.}{\to} c$ as $n \to \infty$. □

Easy to see via the Borel–Cantelli lemmas:

- $U_n \overset{c.c.}{\to} c \implies U_n \overset{a.s.}{\to} c$;
- $\{U_n\}$ independent: $U_n \overset{c.c.}{\to} c \iff U_n \overset{a.s.}{\to} c$.

Hsu–Robbins–Erdős

Theorem, late 40's

$$\sum_{n=1}^\infty P(|S_n| > n\varepsilon) < \infty \quad \forall\, \varepsilon > 0,$$

iff

$$E\,X = 0 \quad \text{and} \quad E\,X^2 = \sigma^2 < \infty.$$

Equivalently:

$$\frac{S_n}{n} \overset{c.c.}{\to} 0 \text{ as } n \to \infty \quad \iff \quad E\,X = 0 \text{ and } E\,X^2 = \sigma^2 < \infty.$$

Remark

The original proof is extremely technical.

Alternative proof – G(1978)

Obvious attempt

$$\sum_{n=1}^\infty P(|S_n| > n\varepsilon) \le \sum_{n=1}^\infty \frac{\sigma^2}{\varepsilon^2 n} = \infty \qquad :-($$

The Kahane-Hoffmann-Jørgensen (KHJ) inequality

$X, X_1, X_2, \ldots, X_n$ symmetric, $S_n = \sum_{k=1}^n X_k$, $x, y > 0$:

$$P(|S_n| > 2x + y) \le P(\max_{1\le k\le n} |X_k| > y) + 4\big(P(|S_n| > x)\big)^2.$$

In particular, in the i.i.d. case with $x = y = n\varepsilon$,

$$P(|S_n| > 3n\varepsilon) \le nP(|X| > n\varepsilon) + 4\big((P(|S_n| > n\varepsilon)\big)^2.$$

$$\implies \sum_{n=1}^\infty P(|S_n| > 3n\varepsilon) \le \sum_{n=1}^\infty nP(|X| > n\varepsilon) + 4\Big(\frac{\sigma^2}{\varepsilon^2}\Big)^2 \sum_{n=1}^\infty \frac{1}{n^2} \qquad :-)$$

An interesting observation

If, in particular,

$$E\,X = 0 \quad \text{and} \quad E\,X^2 = \infty\,,$$

then

$$\begin{cases} P(|S_n| > n\varepsilon \text{ i.o.}) = 0 & \forall\, \varepsilon > 0, \\ \sum_{n=1}^{\infty} P(|S_n| > n\varepsilon) = \infty & \forall\, \varepsilon > 0. \end{cases}$$

Conclusion

Finitely many events occur a.s.

and/but the B–C sum diverges.

Variations and Generalizations

- Marcinkiewicz–Zygmund laws
- Convergence rates
- Precise asymptotics
- Uniform integrability and moment convergence
- Randomly indexed sums
- Weighted sums
- Summation methods; e.g. Cesàro summation
- Other normalizations
- Subsequences
- Arrays
- What happens when $p \nearrow 2$?
- Non-i.i.d. summands
- Multidimensional index sets; random fields
- Vector (Banach space) valued r.v's
- Dependent sequences
- Partial maxima
- Renewal theory (for random walks)
- Records
- The law of the iterated logarithm
- The law of the single logarithm

The Marcinkiewicz–Zygmund strong law

$X_1, X_2, \ldots$ i.i.d., $\quad S_n = \sum_{k=1}^n X_k, \quad 0 < r < 2.$

- If $E|X|^r < \infty$, and $E\,X = 0$ $(1 \le r < 2)$, then

$$\frac{S_n}{n^{1/r}} \stackrel{a.s.}{\to} 0 \quad \text{as} \quad n \to \infty.$$

- If a.s. convergence holds, then

$$E|X|^r < \infty \quad \text{and} \quad E\,X = 0 \quad (1 \le r < 2).$$

Convergence rates LLN — Baum–Katz (1965)

Theorem $\quad r > 0, \quad \alpha > 1/2, \quad \alpha r \ge 1.$

$X, X_1, X_2, \ldots$ i.i.d., $S_n = \sum_{k=1}^n X_k$, $n \ge 1$.

If

$$E|X|^r < \infty \quad \text{and, if } r \ge 1, \quad E\,X = 0,$$

then

$$\sum_{n=1}^{\infty} n^{\alpha r - 2} P(|S_n| > n^{\alpha}\varepsilon) < \infty \quad \text{for all} \quad \varepsilon > 0.$$

And "conversely".

Cesàro summation

Definition $\quad \{x_n,\, n \ge 0\}$ reals

Set, for $\alpha > -1$, $A_0^{\alpha} = 1$ and

$$A_n^{\alpha} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}, \quad n \ge 1.$$

$\{x_n,\, n \ge 0\}$ is *(C, α)-summable* iff

$$\frac{1}{A_n^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} x_k \quad \text{converges as} \quad n \to \infty.$$

In the following $\;0 < \alpha \le 1.$

Strong law $d = 1$

Theorem — (1973, 1988, 1955)

$$\{X_k,\, k \ge 1\} \text{ i.i.d. is a.s. } (C, \alpha)\text{-summable}$$

$$\Longleftrightarrow \quad E|X|^{1/\alpha} < \infty\,,$$

that is,

$$\frac{1}{A_n^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} X_k \stackrel{a.s.}{\to} \mu \quad \text{as} \quad n \to \infty$$

$$\Longleftrightarrow \quad E|X|^{1/\alpha} < \infty \text{ and } E\,X = \mu.$$

Remark $\quad \alpha = 1 \quad \longrightarrow \quad$ SLLN.

Complete convergence $d = 1$

Theorem — G (1993)

$\{X_k,\, k \geq 1\}$ is completely (C, α)-summable to μ,

i.e.,

$$\sum_{n=1}^{\infty} P\Big(\Big|\sum_{k=0}^{n} A_{n-k}^{\alpha-1} X_k - \mu\Big| > A_n^{\alpha}\varepsilon\Big) < \infty \quad \text{for every} \quad \varepsilon > 0\,,$$

if and only if

$$\begin{cases} E|X|^{1/\alpha} < \infty, & \text{for} \quad 0 < \alpha < \frac{1}{2}, \\ E|X|^2 \log^+ |X| < \infty, & \text{for} \quad \alpha = \frac{1}{2}, \\ E|X|^2 < \infty, & \text{for} \quad \frac{1}{2} < \alpha \leq 1, \end{cases}$$

and $E\,X = \mu$.

Some further kinds of strong laws

The law of the iterated logarithm — LIL

$X,\, X_1,\, X_2,\, \dots$ i.i.d. $\quad S_n = \sum_{k=1}^{n} X_k,\; n \geq 1.$

Hartman–Wintner (1941), Strassen (1966)

$$\limsup_{n\to\infty} \,(\liminf_{n\to\infty})\, \frac{S_n}{\sqrt{2n \log\log n}} = +\sigma\ (-\sigma) \quad \text{a.s.}$$

iff $\quad E\,X = 0 \quad$ and $\quad \operatorname{Var} X = \sigma^2 < \infty.$

Windows — delayed sums

$$T_{n,n+k} = \sum_{j=n+1}^{n+k} X_j, \quad k \geq 1, \qquad 0 < \alpha < 1.$$

Convergence in probability is "trivial":

$$T_{n,n+k} \stackrel{d}{=} S_k.$$

SLLN — Chow (1973)

$$\frac{T_{n,n+n^{\alpha}}}{n^{\alpha}} \stackrel{a.s.}{\to} 0 \quad \text{as} \quad n \to \infty$$

$$\Longleftrightarrow$$

$$E|X|^{1/\alpha} < \infty \qquad \text{and} \qquad E\,X = 0\,.$$

The law of the single logarithm — LSL

Window analog to LIL.

Lai (1974)

$$\limsup_{n\to\infty} \,(\liminf_{n\to\infty})\, \frac{T_{n,n+n^{\alpha}}}{\sqrt{2n^{\alpha} \log n}} = \sigma\sqrt{1-\alpha}\ \big(-\sigma\sqrt{1-\alpha}\big) \text{ a.s.}$$

$$\Longleftrightarrow$$

$$E\Big(|X|^{2/\alpha}\,(\log^+|X|)^{-1/\alpha}\Big) < \infty,\ E\,X^2 = \sigma^2\,, E\,X = 0\,.$$

The law of the **single** logarithm

Multiindex

$\{X_{\mathbf{k}},\, \mathbf{k} \in \mathbb{Z}_+^d\}$ i.i.d. $S_{\mathbf{n}} = \sum_{\mathbf{k}\leq\mathbf{n}} X_{\mathbf{k}},\ \mathbf{n} \in \mathbb{Z}_+^d$.

Partial order $\leq$ coordinate-wise.

$\mathbf{n}^{\alpha}$ coordinate-wise α-powers.

$\mathbf{n} \to \infty$ means $n_i \to \infty$ all i, $\quad |\mathbf{n}| = \prod_{i=1}^{d} n_i$.

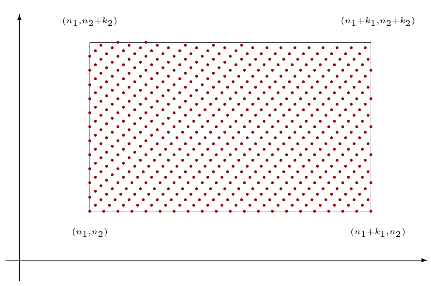
♯ Inequalities for $S_{\mathbf{n}}$ remain "unchanged".

♯ Inequalities for $\max_{\mathbf{k}\leq\mathbf{n}} S_{\mathbf{k}}$ depend on d.

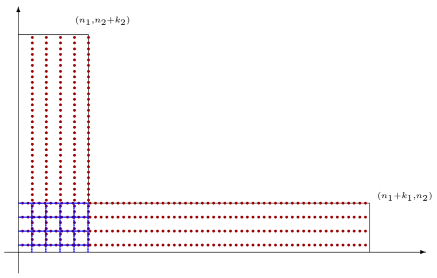
Lévy inequality Symmetric case:

$$P(\max_{\mathbf{k}\leq\mathbf{n}} S_{\mathbf{k}} > x) \leq 2^d P(S_{\mathbf{n}} > x).$$

A typical increment



Overlapping partial sums



Note
$S_{(n_1+k_1,n_2)}$ and $S_{(n_1,n_2+k_2)}$ share $X_{i,j}$, $i \le n_1$, $j \le n_2$.

Problems

- Moments $\longleftrightarrow$ tail probabilities?
- LLN??
- LIL??
- LSL??
- Necessities?
- Other index sets?
- ???

Tail probabilities and moments

$$d = 1: \quad \sum_{n=1}^{\infty} P(|X| > n) < \infty \quad \Longleftrightarrow \quad E\,|X| < \infty.$$

Problem $d \ge 2$:

$$\sum_{\mathbf{n}} P(|X| > |\mathbf{n}|) < \infty \quad \Longleftrightarrow \quad ????.$$

Put

$$d(j) = \mathrm{Card}\,\{\mathbf{k} : |\mathbf{k}| = j\} = o(j^{\delta}), \quad \forall\, \delta > 0\,,$$

$$M(j) = \mathrm{Card}\,\{\mathbf{k} : |\mathbf{k}| \le j\} \sim \frac{j(\log j)^{d-1}}{(d-1)!}\,.$$

Cont'd

Partial summation $\Longrightarrow$:

$$\begin{aligned}
\sum_{\mathbf{n}} P(|X| > |\mathbf{n}|) &= \sum_{j=1}^{\infty} \sum_{|\mathbf{n}|=j} P(|X| > j) \\
&= \sum_{j=1}^{\infty} d(j) P(|X| > j) \\
&\sim E\,M(|X|) \\
&\sim E\,|X|(\log^+ |X|)^{d-1}.
\end{aligned}$$

SLLN, $d \ge 2$

SLLN — Smythe (1973)

$$\frac{S_{\mathbf{n}}}{|\mathbf{n}|} = \frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \le \mathbf{n}} X_{\mathbf{k}} \overset{a.s.}{\to} 0$$

$$\Longleftrightarrow$$

$$E\,M(|X|) < \infty$$

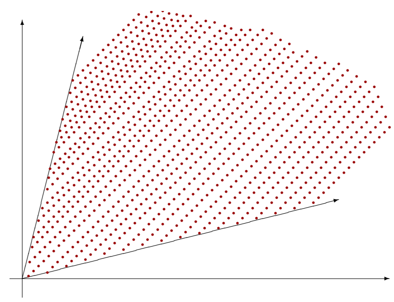
$$\Longleftrightarrow$$

$$E(|X|(\log^+ |X|)^{d-1}) < \infty, \;\; E\,X = 0\,.$$

Remark

WLLN $\Longleftrightarrow$ $nP(|X| > n) \to 0$ (of course ??)

What if the index set is a sector?



$M(x) \sim Cx \Longrightarrow$ SLLN $\Longleftrightarrow$ $E|X| < \infty$ (no logarithms).

Martingales and SLLN

(There exist *four* different kinds of martingales.)

Theorem for "the obvious" martingales

A.s. convergence $\iff$ $L(\log L)^{d-1}$ (dimension).

SLLN $\iff$ $E\,M(|X|) < \infty$ (size of index set)

$$\iff \begin{cases} L(\log L)^{d-1}, & \text{for } \mathbb{Z}_+^d, \\ L, & \text{for the sector.} \end{cases}$$

Martingale proof of SLLN is nice and elegant.

However . . . too weak for the sector

Do not underestimate the beauty
of elementary methods!

The H-R-E-S-B-K theorem, $d \geq 2$ – G(1978)

$\{X_{\mathbf{k}},\, \mathbf{k} \in \mathbf{Z}_+^d\}$ are i.i.d. $S_{\mathbf{n}} = \sum_{\mathbf{k}\leq\mathbf{n}} X_{\mathbf{k}},\, \mathbf{n} \in \mathbf{Z}_+^d$.

$r > 0$, $\alpha > 1/2$, $\alpha r \geq 1$.

If

$$E|X|^r(\log^+|X|)^{d-1} < \infty \quad \text{and} \quad E\,X = 0 \quad (r \geq 1),$$

then

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-2} P(|S_{\mathbf{n}}| > |\mathbf{n}|^{\alpha}\varepsilon) < \infty \quad \text{for all} \quad \varepsilon > 0.$$

And "conversely".

Proof
Similar to the case $d = 1$, exploiting the K–H–J inequality.

Remember !

The remaining part of the talk
is on joint work with
Ulrich Stadtmüller

What if the α's are different?

$$\frac{1}{2} \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_d \leq 1.$$

Define

$$\begin{aligned} p &= \max\{k : \alpha_k = \alpha_1\}; \\ \mathbf{n}^{\alpha} &= (n_1^{\alpha_1}, n_2^{\alpha_2}, \ldots, n_d^{\alpha_d}) \\ |\mathbf{n}^{\alpha}| &= \prod_{k=1}^{d} n_k^{\alpha_k}. \end{aligned}$$

Multi H-R-E-S-B-K (extends G (1978))

$r > 0$, $\alpha_1 > 1/2$, $\alpha_1 r \geq 1$, $\{X_{\mathbf{k}},\, \mathbf{k} \in \mathbf{Z}_+^d\}$ i.i.d. ...

If

$$E|X|^r(\log^+|X|)^{p-1} < \infty \quad \text{and, if } r \geq 1, \quad E\,X = 0,$$

then

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r-2} P(|S_{\mathbf{n}}| > |\mathbf{n}^{\alpha}|\varepsilon) < \infty \quad \text{for all} \quad \varepsilon > 0.$$

And conversely.

Cesàro summation $d = 2$

Definition $\alpha, \beta > 0$ $\{x_{m,n},\, m,n \geq 0\}$ array of reals.

$\{x_{m,n},\, m,n \geq 0\}$ is *(C,α,β)-summable* iff

$$\frac{1}{A_m^{\alpha} A_n^{\beta}} \sum_{m,n} \sum_{k,l=0}^{m,n} A_{n-k}^{\alpha-1} A_{n-l}^{\beta-1} x_{k,l} \quad \text{converges as} \quad m,n \to \infty.$$

In the following $0 < \alpha \leq \beta \leq 1$.

Cesàro summation: Weak law $d = 2$

Theorem $\{X_{k,l},\ k,l \geq 0\}$ i.i.d., *symmetric*

If $nP(|X| > n) \to 0$ as $n \to \infty$, then

$$\frac{1}{A_m^\alpha A_n^\beta} \sum_{k,l=0}^{m,n} A_{m-k}^{\alpha-1} A_{n-l}^{\beta-1} X_{k,l} \xrightarrow{p} 0 \quad \text{as} \quad m,n \to \infty.$$

- Proof: Truncation; set

$$Y_{k,l}^{m,n} = A_{m-k}^{\alpha-1} A_{n-l}^{\beta-1} X_{k,l} I\{|X_{k,l}| \leq A_m^\alpha A_n^\beta\}.$$

- Use weak convergence criterion.

Note the Feller condition.

Cesàro summation: Strong law $d = 2$

Theorem

$\{X_{k,l},\ k,l \geq 0\}$ is a.s. (C, α, β)-summable, viz.,

$$\frac{1}{A_m^\alpha A_n^\beta} \sum_{k,l=0}^{m,n} A_{m-k}^{\alpha-1} A_{n-l}^{\beta-1} X_{k,l} \xrightarrow{a.s.} \mu \quad \text{as} \quad n,m \to \infty$$

if and only if

$$\begin{cases} E|X|^{\frac{1}{\alpha}}, & \text{for} \quad 0 < \alpha < \beta \leq 1, \\ E|X|^{\frac{1}{\alpha}} \log^+ |X|, & \text{for} \quad 0 < \alpha = \beta \leq 1. \end{cases}$$

and $E\,X = \mu$.

Proof Kolmogorov 3-series thm + Kronecker (d=2)

Cesàro summation: Complete convergence $d = 2$

Theorem $\{X_{k,l},\ k,l \geq 0\}$ is

completely (C, α, β)-summable to μ, i.e.,

$$\sum_{m,n} P\Big(\Big|\sum_{k,l=0}^{m,n} A_{m-k}^{\alpha-1} A_{n-l}^{\beta-1} X_{k,l} - \mu\Big| > A_m^\alpha A_n^\beta \varepsilon\Big) < \infty$$

for every $\varepsilon > 0$, $\iff$

$$\begin{cases} E|X|^{\frac{1}{\alpha}}, & \text{for} \quad 0 < \alpha < 1/2,\ \alpha < \beta \leq 1, \\ E|X|^{\frac{1}{\alpha}} \log^+ |X|, & \text{for} \quad 0 < \alpha = \beta < \frac{1}{2}, \\ E|X|^2 \log^+ |X|, & \text{for} \quad \frac{1}{2} < \alpha \leq \beta \leq 1, \\ E|X|^2 (\log^+ |X|)^2, & \text{for} \quad \alpha = \frac{1}{2} < \beta \leq 1, \\ E|X|^2 (\log^+ |X|)^3, & \text{for} \quad \alpha = \beta = \frac{1}{2}. \end{cases}$$

and $E\,X = \mu$.

Delayed sums: LSL — Different α's

Theorem

$\{X_\mathbf{k},\ \mathbf{k} \in \mathbb{Z}_+^d\}$ i.i.d., $E\,X = 0$, $\mathrm{Var}\,X = \sigma^2$.

$$0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_d < 1 \quad \text{and} \quad p = \max\{k : \alpha_k = \alpha_1\}.$$

If

$$E|X|^{2/\alpha_1} (\log^+ |X|)^{p-1-1/\alpha_1} < \infty,$$

then

$$\limsup_{\mathbf{n} \to \infty} \frac{T_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}}{\sqrt{2|\mathbf{n}^\alpha| \log |\mathbf{n}|}} = \sigma\sqrt{1 - \alpha_1} \quad \text{a.s.}$$

And "conversely". □

Delayed sums: A degenerate case $\alpha = 0$

$$0 = \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_d < 1$$
$$q = \max\{k : \alpha_k = 0\} \qquad r = \max\{k : \alpha_k = \alpha_{q+1}\}.$$

If

$$E|X|^{2/\alpha_{q+1}} (\log^+ |X|)^{r-q-1-1/\alpha_{q+1}} < \infty,$$

then

$$\limsup_{\mathbf{n} \to \infty} \frac{T_{\mathbf{n},\mathbf{n}+\mathbf{n}^\alpha}}{\sqrt{2|\mathbf{n}^\alpha| \log |\mathbf{n}|}} = \sigma\sqrt{1 - \alpha_{q+1}} \quad \text{a.s.}$$

And conversely.

In particular $q = 0$, $r = d$ and $q = 0$, $r = p$.

Sequences: A boundary problem

Finite variance, mean 0 $\iff$ LIL;

$$\limsup_{n \to \infty} \frac{S_n}{\sqrt{n \log\log n}} = \sigma\sqrt{2}$$

$E|X|^r < \infty$ $(0 < r < 2,\ E\,X = 0,\ 1 \leq r < 2)$ $\iff$ MZ;

$$\frac{S_n}{n^{1/r}} \xrightarrow{a.s.} 0 \quad \text{as} \quad n \to \infty.$$

Question

What if, say, $EX^2/\log^+|X| < \infty$??

Analogs for arrays and Cesàro summation.

Examples

♢ Suppose that $E\,X^2/\log^+|X| < \infty$. Then

$$\frac{S_n}{\sqrt{n \log n \log\log n}} \stackrel{a.s.}{\to} 0 \quad \text{as} \quad n \to \infty.$$

♢ Suppose that $E\,X^2/(\log^+\log^+|X|)^{\alpha-1} < \infty$. Then

$$\frac{S_n}{\sqrt{n(\log\log n)^{\alpha}}} \stackrel{a.s.}{\to} 0 \quad \text{as} \quad n \to \infty.$$

♢ Suppose that $E\,X^2/(\log^+\log^+\log^+|X|)^{\beta} < \infty$. Then

$$\frac{S_n}{\sqrt{n \log\log n (\log\log\log n)^{\beta}}} \stackrel{a.s.}{\to} 0 \quad \text{as} \quad n \to \infty\,.$$

Cesàro summation

Recall: $0 < \alpha \le 1$, $Y, Y_1, Y_2, \ldots$ are i.i.d. r. v's., and

$$S_n = \sum_{j=0}^{n} A_{n-j}^{\alpha-1} Y_j.$$

The central limit theorem for Cesàro summation:

$$\frac{S_n}{\sigma\sqrt{\frac{\log n}{\pi}}} \stackrel{d}{\to} N(0,1) \quad \text{as} \quad n \to \infty \quad \text{for} \quad \alpha = 1/2,$$

$$\frac{S_n}{\frac{\sigma}{\Gamma(\alpha)}\sqrt{\frac{n^{2\alpha-1}}{2\alpha-1}}} \stackrel{d}{\to} N(0,1) \quad \text{as} \quad n \to \infty \quad \text{for} \quad 1/2 < \alpha \le 1.$$

$\alpha = 1 \quad \longleftrightarrow \quad$ "usual" CLT

Examples; $1/2 < \alpha < 1$

♣ If $E\Big(\frac{|Y|}{\log^+|Y|}\Big)^{2/(2\alpha-1)} < \infty$, then

$$\frac{S_n}{\sqrt{n^{2\alpha-1}(\log n)^2}} \stackrel{a.s.}{\to} 0 \quad \text{as} \quad n \to \infty,$$

♣ If $E\Big(\frac{Y^2}{\log^+|Y|(\log^+\log^+|Y|)^{\beta}}\Big)^{1/(2\alpha-1)} < \infty$, then

$$\frac{S_n}{\sqrt{n^{2\alpha-1}\log n(\log\log n)^{\beta}}} \stackrel{a.s.}{\to} 0 \quad \text{as} \quad n \to \infty.$$

Examples; $\alpha = 1/2$

Let $\delta > 0$ be arbitrary.

♡ If $\inf\{t : E\,e^{t|Y|/\log^+|Y|} = \infty\} > 0$, then

$$\frac{S_n}{\log n \log\log n\,(\log\log\log n)^{\delta}} \stackrel{a.s.}{\to} 0 \quad \text{as} \quad n \to \infty.$$

♡ If $\inf\{t : E\,e^{t\sqrt{|Y|}} = \infty\} > 0$, then

$$\frac{S_n}{(\log n)^2\,(\log\log n)^{\delta}} \stackrel{a.s.}{\to} 0 \quad \text{as} \quad n \to \infty.$$

References

http://www2.math.uu.se/~allan/berlin2017ref.pdf

Thank you